

Flavored $\mathcal{N} = 4$ SYM – a highly entangled quantum liquid

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We study $\mathcal{N} = 4$ SYM theory coupled to fundamental $\mathcal{N} = 2$ hypermultiplets in a state of finite charge density. The setup can be described holographically as a configuration of D3 and D7 branes with a non-trivial worldvolume gauge field on the D7. The phase has been identified as a new form of quantum liquid, where certain properties are those of a Fermi liquid while others are clearly distinct. We focus on the entanglement among the flavors, as quantified by the entanglement entropy. The expectation for a Fermi liquid would be a logarithmic enhancement of the area law, but we find a more drastic enhancement instead. The leading contributions are volume terms with a non-trivial shape dependence, signaling extensive entanglement among the flavors. At finite temperature these correlations are confined to a region of size given by the inverse temperature.

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I. INTRODUCTION

Exploring the possible quantum phases of matter has become a significant part of modern physics. Established models like Fermi liquid theory do not account for the variety of phases realized in condensed matter systems, and new concepts are called for. Via the AdS/CFT dualities [1–3] string theory has become a useful tool to map out the landscape of quantum phases of matter, and in this work we study one specific example. We focus on $\mathcal{N} = 4$ supersymmetric Yang-Mills theory (SYM) coupled to $\mathcal{N} = 2$ flavor hypermultiplets in a state of finite charge density. In the sense of being a translationally invariant zero-temperature state of finite charge density, this is indeed a quantum liquid. In fact, a rather interesting one [4]: despite a zero-sound mode with the properties expected for a Fermi liquid, other properties like the specific heat clearly characterize it as a non-Fermi liquid. It also exhibits a degenerate ground state, signaled by a non-vanishing

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entropy density at zero temperature, which is a feature often associated with topological order [5, 6],

The study of entanglement has emerged as a powerful diagnostic to detect and characterize different phases of matter, which in particular includes non-Fermi liquid behavior [7]. Entanglement entropy as the most prominent quantification is used, e.g., to study quantum critical phenomena [8] and to detect topological order [9, 10]. The usual definition in quantum field theory rests on a geometric split into two subsystems A and B at a constant time. For a global state described by a density operator ρ , it is then defined as the von-Neumann entropy of the reduced density operator ρ_A for, say, the subsystem A ¹

$$\mathcal{S}_{\text{EE}} = -\text{tr } \rho_A \log \rho_A, \quad \rho_A = \text{tr}_B \rho. \quad (1)$$

We use it here to study correlations among the charge carriers in flavored $\mathcal{N} = 4$ SYM to better understand the phase of finite charge density. Our results may also be of more general interest, as many of the investigations of entanglement entropy have focused on the vacuum and analytical results for more general states are still rather scarce.

The actual evaluation of (1) for QFTs using the replica trick is non-trivial, even for free theories. Fortunately, we can exploit the fact that the computation is greatly simplified in the dual gravity description as obtained from AdS/CFT. In that context, the $\mathcal{N} = 4$ SYM coupled to fundamental $\mathcal{N} = 2$ flavor hypermultiplets corresponds to a configuration of intersecting D3/D7 branes [12]. The finite charge density is realized by a non-trivial background profile for the D7 worldvolume gauge field corresponding to the diagonal U(1) of the flavor symmetry group [13–16]. The quenched approximation on the field theory side corresponds to weakly backreacting D7 branes. Computing entanglement entropies holographically simply amounts to evaluating the area of a minimal surface in AdS [17, 18], which is strikingly simpler than the corresponding field-theory calculation. Nevertheless, in the present context even that becomes tough. To obtain the flavor contribution to the entanglement entropy, one would first have to compute the backreaction of the D7 branes with the worldvolume gauge field on the background geometry created by the D3 branes. For D7 branes homogeneously smeared over the compact part of the background geometry that backreaction has been obtained in [19, 20]. In the perturbed geometry one would then have to find the appropriate minimal surface and compute its area. For the case of massive flavors at zero density this could be done analytically in [21].

Recently, two holographic methods have been developed [22, 23], which avoid dealing with the backreaction explicitly and allow us to make progress more easily. For the calculations here we start from the method of [22]. It only needs the linearized backreaction of an effective source on the AdS₅ part of the background geometry created by the D3 branes. The change in the minimal area can then be obtained as an integral over the minimal surface in the unperturbed geometry. To obtain results in a decent closed form for the state of finite charge density, we further exploit that representation. This eventually allows us to avoid an explicit computation of the backreaction altogether.

The paper is organized as follows. We review the holographic setting in Section II and set up Einstein’s equations for the effective backreaction of the D7 branes on the non-compact part of the geometry. In Section III we calculate the entanglement entropies for a disc and a strip in closed form as simple one-dimensional integrals. We discuss the limits of small and large entangling surfaces in Section IV. We recover the universal entanglement temperature predicted for small regions and argue that the specific form of the volume terms found for large regions signals extensive entanglement. In Section V we consider finite temperature and discuss the transition to the thermal entropy. A summary and discussion of our results is given in the final Section VI.

¹ For gauge theories, the geometric split does not yield a tensor decomposition of the Hilbert space, but an entanglement entropy can still be defined [11].

II. D3/D7 WITH WORLDVOLUME GAUGE FIELD: EFFECTIVE EINSTEIN EQUATIONS

We start with a review of the setup used in [13–16] to holographically describe flavored $\mathcal{N} = 4$ SYM at finite charge density. The discussion of the D3/D7 system straightforwardly generalizes to probe branes with a $p + 1$ dimensional worldvolume in AdS_{d+1} times an internal space. For the D3/D7 system the corresponding $\text{AdS}_5 \times S^5$ space is created by the D3 branes and the D7 branes are embedded as probes. More generally, we consider a black-brane background which is a product of an asymptotically- AdS_{d+1} black hole with metric

$$ds^2 = \frac{L^2}{z^2} \left[\frac{dz^2}{b(z)} - b(z)dt^2 + d\vec{x}^2 \right], \quad b(z) = 1 - \frac{z^d}{z_h^d}, \quad z_h = \frac{d}{4\pi T}, \quad (2)$$

and an internal space X . In this background we consider probe Dp -branes wrapping the entire AdS_{d+1} part. The Dirac-Born-Infeld (DBI) action governing the dynamics of the Dp probe involves a $U(N_f)$ gauge field, of which we only need the diagonal $U(1)$. To study the dual theory at finite density, we add a non-trivial profile for $A_t(z)$. The DBI action in mostly-plus signature then reads

$$S_{\text{brane}} = -N_f T_p \int d^{p'+1}y \sqrt{-\det(\gamma + 2\pi\alpha' F)}. \quad (3)$$

The relation of N_f and T_p to field-theory quantities is given, e.g., in [4]. For massless flavors, the branes wrap a constant part of the internal space and we can just integrate that out. This yields a factor of the volume in the internal space, V_X , and we arrive at

$$S_{\text{brane}} = -T_0 \int d^{d+1}y_s \sqrt{-\det(\gamma_s + 2\pi\alpha' A'_t dz \wedge dt)}, \quad (4)$$

where $T_0 = N_f T_p V_X$. We drop the subscript s in the following. The action does not depend on A_t itself, only on the radial derivative. Hence we get a conserved quantity q , which corresponds to the charge density in the dual field theory. We absorb a factor $2\pi\alpha'$ and define it by

$$2\pi\alpha' T_0 q := \frac{\delta S_{\text{brane}}}{\delta A'_t} = T_0 \sqrt{-\gamma} \frac{(2\pi\alpha')^2 F^{tz}}{\sqrt{1 + (2\pi\alpha')^2 F^{zt} F_{zt}}}. \quad (5)$$

This can be solved straightforwardly for A'_t . We now turn to setting up Einstein's equations for the effective backreaction of (4) on the AdS_5 part of the background geometry, (2). For more detailed discussions of the D3/D7 setup we refer to [13–16]. The effective energy-momentum tensor for (4) reads

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S_{\text{brane}}}{\delta g_{\mu\nu}} = -T_0 \sqrt{\det[g^{-1}(\gamma + 2\pi\alpha' F)]} (\gamma + 2\pi\alpha' F)^{-1 \{\mu\nu\}}. \quad (6)$$

Here we only consider the effective theory after integrating out the internal space. As we will review in the next section, this has been shown to be sufficient to capture the leading order correction to the entanglement entropy [22]. The indices of $(\gamma + 2\pi\alpha' F)^{-1}$ are symmetrized since $\delta g_{\mu\nu}$ is symmetric by construction. The only non-vanishing components are

$$T^{zz} = -T_0 g^{zz} \sqrt{1 + q^2 z^{2d-2}}, \quad T^{tt} = -T_0 g^{tt} \sqrt{1 + q^2 z^{2d-2}}, \quad T^{ij} = -\frac{T_0 g^{ij}}{\sqrt{1 + q^2 z^{2d-2}}}. \quad (7)$$

Anticipating that the linearly backreacted geometry will be asymptotically AdS_{d+1} with the same radius of curvature as in the zero-density zero-temperature case, we choose the ansatz

$$g + \delta g = \left(1 + \frac{t_0}{d(d-1)}\right) g_{zz} dz^2 + (1 + t_0 h(z)) g_{tt} dt^2 + (1 + t_0 j(z)) g_{ij} dx^i dx^j, \quad (8)$$

where $t_0 = \kappa T_0$ and $\kappa = 16\pi G$. We have also set $L = 1$. The non-trivial components of Einstein's equations are the zz , tt and ii components. They only involve derivatives of j and h , and to discuss their solutions we define \tilde{j} and \tilde{h} by

$$\tilde{j}(z) = z^{1-d} j'(z) , \quad \tilde{h}(z) = z^{1-d} h'(z) . \quad (9)$$

The equations then simplify quite a bit and read

$$(d-1)z^d \left(b(z)\tilde{h}(z) + (d-1)b(z)\tilde{j}(z) - \frac{1}{2}zb'(z)\tilde{j}(z) \right) = \sqrt{1+q^2z^{2d-2}} - 1 , \quad (10a)$$

$$(d-1)z^{d+1}\sqrt{b(z)} \left(\sqrt{b(z)}\tilde{j}(z) \right)' = 1 - \sqrt{1+q^2z^{2d-2}} , \quad (10b)$$

$$z^{d+1} \left(b(z)\tilde{h}'(z) + \frac{3}{2}b'(z)\tilde{h}(z) + (d-2)(b(z)\tilde{j}(z))' \right) = 1 - \frac{1}{\sqrt{1+q^2z^{2d-2}}} . \quad (10c)$$

We can verify that these equations can indeed be solved with our ansatz as follows. The first equation can be solved for \tilde{h} in terms of \tilde{j} . Using the result in the third equation then reproduces the second one. Thus, once (10b) is solved for \tilde{j} , this can indeed be extended to a full solution. We impose regularity at the horizon by demanding $b\tilde{h}$ and $b\tilde{j}$ to vanish as $z \rightarrow z_h$. Using this in either (10a) or (10b) both yields

$$\tilde{j}(z_h) = \frac{2}{d(d-1)}z_h^{-d} \left(\sqrt{1+q^2z_h^{2d-2}} - 1 \right) . \quad (11)$$

We focus on $T = 0$ in the next sections, and come back to $T > 0$ in Sec. V. This means $b(z) = 1$ and the equations simplify further. The condition (11) reduces to the demand that $\tilde{j}(z)$ vanishes as $z \rightarrow \infty$. Using that, we integrate both sides of (10b) on $[z, \infty)$, and find

$$\tilde{j}(z) = -\frac{1}{d(d-1)z^d} + \frac{q}{(d-1)z} {}_2F_1 \left(-\frac{1}{2}, \frac{1}{2d-2}, \frac{2d-1}{2d-2}, -\frac{1}{z^{2d-2}q^2} \right) . \quad (12)$$

We did not find a closed expression for j , which is obtained by integrating \tilde{j} with (9). We will therefore use integration by parts later to express the area in terms of \tilde{j} . To this end we need the near-boundary expansion of j . This can be obtained from the expansion of \tilde{j} , for which we find

$$\tilde{j}(z) = \frac{\alpha(d)}{d} q^{d/(d-1)} + \mathcal{O}(z^{d-2}) , \quad \alpha(d) = \frac{\Gamma(\frac{1}{2} - \frac{1}{2d-2})\Gamma(1 + \frac{1}{2d-2})}{\sqrt{\pi}} . \quad (13)$$

Integrating this further yields the near-boundary expansion of j

$$j(z) = c_2 + c_1 z^d + \mathcal{O}(z^{2d-2}) , \quad c_1 = \frac{\alpha(d)}{d^2} q^{d/(d-1)} , \quad (14)$$

where the subleading parts are fixed and the remaining freedom is in the choice of c_2 . Eq. (10a) only fixes the derivative of h in terms of the derivative of j , so there is an analogous free constant in h . To asymptotically get a Poincaré-AdS metric, we fix $h(0) = c_2 = 1/[d(d-1)]$. This ensures that the perturbed geometry describes the perturbed dual CFT in the vacuum state, i.e. we do not want to source T_{tt} and T_{ii} . Using (10a) to relate the $\mathcal{O}(z^d)$ parts of h and j , we find

$$h(z) = c_2 - (d-1)c_1 z^d + \mathcal{O}(z^{2d-2}) , \quad c_2 = \frac{1}{d(d-1)} . \quad (15)$$

For the discussion of entanglement thermodynamics below we will also need the energy density $\langle T_{tt} \rangle_{\text{CFT}}$ for the CFT at finite charge density. It can be derived straightforwardly from the

perturbed bulk metric, using the one-point function of the renormalized CFT energy-momentum tensor as computed, e.g., in [24]. It reads

$$\langle T_{\mu\nu} \rangle_{\text{CFT}} = \frac{dL'^{d-1}}{16\pi G} g_{\mu\nu}^{(d)} , \quad (16)$$

where L' is the radius of curvature of the perturbed solution, $L' = L(1 + t_0 c_2/2) + \mathcal{O}(t_0^2)$. With (8) and the near-boundary expansions (14), (15), we thus find, to linear order in t_0 ,

$$\langle T_{tt} \rangle_{\text{CFT}} = d(d-1)T_0 c_1 , \quad \langle T_{ii} \rangle_{\text{CFT}} = dT_0 c_1 . \quad (17)$$

We note that, using $\langle T_{ii} \rangle_{\text{CFT}} = p$, we reproduce the thermodynamic pressure $p = -(\partial\Omega/\partial V_{d-1})_{\mu,T}$ obtained from the potential Ω given in [4], validating our backreaction.

We close the section with a comment on the validity of the linearized approximation. The source term on the right hand side of Einstein's equations (10) grows unboundedly in the IR. At zero temperature we can therefore not trust the linearized approximation for the IR limit $z \rightarrow \infty$. This point has been discussed in detail in [20, 25]. For the discussion of the entanglement entropy below, we will be interested in regions of large but finite extent, such that the corresponding minimal surfaces have a finite extension into the bulk. We thus do not need the actual IR limit and the backreaction in the region probed by the minimal surfaces is small as long as t_0 is sufficiently small or, equivalently, if for a given fixed small t_0 the minimal surface is not parametrically large in $1/t_0$.

III. ENTANGLEMENT ENTROPY AT ZERO TEMPERATURE

In the following we compute the entanglement entropy at zero temperature. To give the results in a clear form, we expand it as

$$\mathcal{S}_{\text{EE}} = \mathcal{S}_{\text{EE}}^{(0)} + \mathcal{S}_{\text{EE}}^{(1)} + \mathcal{O}(t_0^2) , \quad \mathcal{S}_{\text{EE}}^{(1)} = \mathcal{S}_{\text{EE},q=0}^{(1)} + \Delta\mathcal{S}_{\text{EE}}^{(1)} . \quad (18)$$

That is, $\mathcal{S}_{\text{EE}}^{(i)}$ denotes the $\mathcal{O}(t_0^i)$ contribution, but we have not extracted the power of t_0 explicitly. Moreover, we have isolated the renormalized entanglement entropy $\Delta\mathcal{S}_{\text{EE}}$, obtained by subtracting off the entanglement entropy in the vacuum state. The leading contribution to the renormalized entropy is $\Delta\mathcal{S}_{\text{EE}}^{(1)}$. Following [22], the $\mathcal{O}(t_0)$ change in the entanglement entropy due to the backreaction of the flavor branes can be expressed as an integral over the minimal surface in the unperturbed geometry

$$\mathcal{S}_{\text{EE}}^{(1)} = \frac{1}{4G} \int \frac{1}{2} T_{\text{min}}^{\mu\nu} \delta g_{\mu\nu} , \quad T_{\text{min}}^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta A_{\text{min}}^{(0)}}{\delta g_{\mu\nu}} , \quad (19)$$

where $A_{\text{min}}^{(0)}$ denotes the area of the minimal surface in the unperturbed geometry. This would usually involve an 8-dimensional minimal surface and the backreaction in the entire 10-dimensional spacetime. Nicely enough, though, as shown in [22] using a detour via a double-integral formula, the details of the internal space can be subsumed into an effective source on the AdS part. In our setting this is just (4), and with the discussion of the effective backreaction in the previous section, we can now evaluate (19) for a spherical region and a strip.

A. Disc

To compute the $\mathcal{O}(t_0)$ entanglement entropy for the spherical region $A : |\vec{x}| \leq \ell$, we switch to spherical coordinates such that $g_{ij} dx^i dx^j = z^{-2} (dr^2 + r^2 d\Omega_{d-2}^2)$. The original minimal surface can

then be parametrized by $z = \ell s$, $r = \ell \sqrt{1 - s^2}$ and is the hyperbolic space \mathbb{H}^{d-1} . This yields the induced metric on the minimal surface

$$\gamma_{\min} = \frac{L^2}{s^2} \left(\frac{ds \otimes ds}{1 - s^2} + (1 - s^2) g_{\mathbb{S}^{d-2}} \right). \quad (20)$$

The variation of the minimal area with respect to the bulk metric for $L = 1$ yields

$$T_{\min} = s^2 \ell^2 \left(\sqrt{1 - s^2} \partial_z - s \partial_r \right) \otimes \left(\sqrt{1 - s^2} \partial_z - s \partial_r \right) + g_{\mathbb{S}^{d-2}}^{ij} \frac{s^2}{1 - s^2} \partial_i \otimes \partial_j. \quad (21)$$

With the ansatz (8) for the perturbed bulk metric, the change in the minimal area, (19), becomes

$$\mathcal{S}_{\text{EE}}^{(1)} = \frac{t_0 V_{\mathbb{S}^{d-2}}}{4G} \int_{\epsilon/\ell}^1 ds \frac{(1 - s^2)^{\frac{d-3}{2}}}{2s^{d-1}} \left((s^2 + d - 2) j(\ell s) + \frac{1 - s^2}{d(d-1)} \right). \quad (22)$$

To compute the separate parts according to (18), we note that $j(\ell s)|_{q=0} = c_2$. This yields

$$\mathcal{S}_{\text{EE},q=0}^{(1)} = \frac{t_0}{2d} \frac{V_{\mathbb{S}^{d-2}}}{4G} \int_{\epsilon/\ell}^1 ds s^{1-d} (1 - s^2)^{\frac{d-3}{2}} = \frac{t_0}{2d} \frac{V_{\mathbb{H}^{d-1}}}{4G}. \quad (23)$$

In the last equality we have introduced the regularized volume of the hyperbolic space which is the original minimal surface. This nicely reproduces the zero-density results of [22, 26]. The remaining part is the renormalized entanglement entropy, which becomes

$$\Delta \mathcal{S}_{\text{EE}}^{(1)} = \frac{t_0 V_{\mathbb{S}^{d-2}}}{4G} \int_{\epsilon/\ell}^1 ds \frac{(1 - s^2)^{\frac{d-3}{2}}}{2s^{d-1}} (s^2 + d - 2) (j(\ell s) - c_2). \quad (24)$$

To evaluate it without having to solve for the backreaction explicitly, we use integration by parts, which yields

$$\frac{4G}{t_0 V_{\mathbb{S}^{d-2}}} \Delta \mathcal{S}_{\text{EE}}^{(1)} = \frac{\ell^{d-2}}{2\epsilon^{d-2}} \left[1 - \frac{\epsilon^2}{\ell^2} \right]^{\frac{d-1}{2}} (j(\epsilon) - c_2) + \frac{\ell^d}{2} \int_{\epsilon/\ell}^1 ds s (1 - s^2)^{\frac{d-1}{2}} \tilde{j}(\ell s). \quad (25)$$

In the second term we have expressed $j'(\ell s)$ in terms of $\tilde{j}(\ell s)$ using (9). From the near-boundary expansion (14), we see that $j(\epsilon) - c_2 = \mathcal{O}(\epsilon^d)$, such that the first term vanishes and we are only left with the second one. The integrand is finite for $s \rightarrow 0$, such that we can equivalently set the lower bound of integration to zero. With \tilde{j} given in (12) our final result then becomes

$$\Delta \mathcal{S}_{\text{EE}}^{(1)} = \frac{t_0}{8G} V_{\mathbb{S}^{d-2}} \int_0^1 ds s (1 - s^2)^{\frac{d-1}{2}} \ell^d \tilde{j}(\ell s). \quad (26)$$

We note that $\ell^d \tilde{j}(\ell s)$ and thus $\Delta \mathcal{S}_{\text{EE}}^{(1)}$ depend on q and ℓ only through the dimensionless combination $q \ell^{d-1}$.

B. Strip

We now turn to the entanglement entropy for a strip defined by $|x^1| \leq \ell/2$ at $t=0$. Using Cartesian coordinates, $g_{ij} dx^i dx^j = z^{-2} d\vec{x}^2$, the corresponding minimal surface can be parametrized

by z , $x^1 = x^1(z)$ and x^{i_0} , $i_0 = 2, \dots, d-1$. As discussed already in [17], extremizing the area in AdS_{d+1} leads to

$$\frac{dx^1}{dz} = \pm \frac{1}{\sqrt{(\ell_\star/z)^{2d-2} - 1}}, \quad \ell_\star = \frac{\ell}{2\sqrt{\pi}} \frac{\Gamma(\frac{1}{2d-2})}{\Gamma(\frac{d}{2d-2})}, \quad (27)$$

where ℓ_\star marks how far the minimal surface extends into the bulk. The induced metric then is

$$\gamma_{\min} = \frac{L^2}{z^2} \left(\frac{dz \otimes dz}{1 - (z/\ell_\star)^{2d-2}} + \sum_{i=2}^{d-1} dx^i \otimes dx^i \right). \quad (28)$$

Since δg is diagonal, we only need the diagonal elements of T_{\min} , and they evaluate to

$$T_{\min}^{zz} = \gamma_{\min}^{zz}, \quad T_{\min}^{11} = \frac{\ell_\star^2}{L^2} \frac{z^{2d}}{\ell_\star^{2d}}, \quad T_{\min}^{i_0 j_0} = \gamma_{\min}^{i_0 j_0}. \quad (29)$$

The $\mathcal{O}(t_0)$ entanglement entropy correction (19) with $L = 1$ then becomes

$$\mathcal{S}_{\text{EE}}^{(1)} = \frac{t_0 V_{d-2}}{4G} \int_\epsilon^{\ell_\star} dz \sqrt{\gamma_{\min}} \left[\frac{1 - (z/\ell_\star)^{2d-2}}{d(d-1)} + j(z) \left(\frac{z^{2d-2}}{\ell_\star^{2d-2}} + d - 2 \right) \right]. \quad (30)$$

The $(d-2)$ -dimensional volume of the strip in the transverse directions is denoted by V_{d-2} , and the integration over z from ϵ to ℓ_\star and back gives a factor 2. We now perform a change of variables to $v = z/\ell_\star$ and slightly rearrange the terms in square brackets, to arrive at

$$\mathcal{S}_{\text{EE}}^{(1)} = \frac{t_0 V_{d-2}}{4G} \ell_\star^{2-d} \int_{\epsilon/\ell_\star}^1 dv \frac{v^{1-d}}{\sqrt{1 - v^{2d-2}}} \left[\frac{1}{d} + (j(v\ell_\star) - c_2) (v^{2d-2} + d - 2) \right]. \quad (31)$$

The first term in square brackets yields the zero-density result, and the second one the renormalized entropy. As a consistency check and for later reference we also evaluate the former, which yields

$$\mathcal{S}_{\text{EE}, q=0}^{(1)} = \frac{t_0}{2d} \frac{1}{4G} \frac{V_{d-2}}{d-2} \left[\frac{2}{\epsilon^{d-2}} - \ell_\star^{2-d} \left(2\sqrt{\pi} \frac{\Gamma(\frac{d}{2(d-1)})}{\Gamma(\frac{1}{2(d-1)})} \right)^{d-1} \right]. \quad (32)$$

This is the result found in [17] up to a factor $t_0/(2d)$, in accordance with [22, 26]. We can now turn to the finite-density contribution, given by the second term in the square brackets in (31). Using integration by parts again, we obtain

$$\frac{4G}{t_0 V_{d-2}} \Delta \mathcal{S}_{\text{EE}}^{(1)} = - \left[(j(v\ell_\star) - c_2) v^{2-d} \sqrt{1 - v^{2d-2}} \right]_{\epsilon/\ell_\star}^1 + \ell_\star^2 \int_{\epsilon/\ell_\star}^1 dv \tilde{j}(v\ell_\star) v \sqrt{1 - v^{2d-2}}. \quad (33)$$

Since $j(z) - c_2 = \mathcal{O}(z^d)$, the first term does not contribute as $\epsilon \rightarrow 0$, and we are left with the second one. The lower bound of integration again can be set to zero since the integrand is finite. With \tilde{j} given in (12) and ℓ_\star in (27) the final result reads

$$\Delta \mathcal{S}_{\text{EE}}^{(1)} = \frac{t_0}{4G} V_{d-2} \ell_\star^2 \int_0^1 dv \tilde{j}(v\ell_\star) v \sqrt{1 - v^{2d-2}}. \quad (34)$$

Due to the factor V_{d-2} , the entanglement entropy for the strip is not a function of $q\ell^{d-1}$ alone, but we note that the rescaled entropy $\ell^{d-2} \Delta \mathcal{S}_{\text{EE}}^{(1)}$ is.

IV. ENTANGLEMENT THERMODYNAMICS AND SHAPE-DEPENDENT VOLUME TERMS

In this section we first derive the behavior of the entanglement entropy for the limits of small and large entangling surfaces, which means the narrow and wide limits for the strip. We then discuss the behavior for small regions in the context of the 1st law of entanglement thermodynamics, and relate the result for large regions to the entropy of the global state. The expressions for small regions can be obtained straightforwardly by expanding (26) and (34) for small $q\ell^{d-1}$. The expansion of \tilde{j} for small argument is given in (13), and the entanglement entropies (26) and (34) then evaluate to

$$\Delta\mathcal{S}_{\text{EE}}^{(1)} = \frac{t_0}{2d} \frac{V_{S^{d-2}}}{4G} \frac{\alpha(d)}{d+1} q^{\frac{d}{d-1}} \ell^d, \quad \Delta\mathcal{S}_{\text{EE}}^{(1)} = \frac{t_0}{2d} \frac{V_{d-2}}{4G} \ell_\star^2 q^{\frac{d}{d-1}} \frac{\sqrt{\pi}\alpha(d)\Gamma(\frac{1}{d-1})}{(d+1)\Gamma(\frac{1}{2} + \frac{1}{d-1})}, \quad (35)$$

for the sphere and the strip, respectively.

For large entangling surface the analogous expansion is less straightforward, and we work it out in some detail for the sphere. The point is that, no matter how large $q\ell^{d-1}$ is, $q(s\ell)^{d-1}$ is always small in some part of the integration region, so we can not simply expand the integrand in (26). To still obtain a systematic expansion we first perform one more integration by parts in (26), such that the resulting integral only involves \tilde{j}' . Using $\tilde{j}(0) = dc_1$ and (10b) to eliminate \tilde{j}' , this yields

$$\Delta\mathcal{S}_{\text{EE}}^{(1)} = \frac{t_0 V_{S^{d-2}}}{8(d+1)G} \left[\ell^d dc_1 + \int_0^1 ds \frac{(1-s^2)^{\frac{d+1}{2}}}{(d-1)s^{d+1}} \left(1 - \sqrt{1 + q^2(\ell s)^{2d-2}} \right) \right]. \quad (36)$$

We now split the integral using $[0, 1] = [0, \eta] \cup [\eta, 1]$ with $\eta \ll 1$. On $[\eta, 1]$, s is bounded from below by η , so we can expand the square root in (36). On $[0, \eta]$ s is small itself, so we can expand $(1-s^2)^{(d+1)/2}$. The resulting integrals in both regions can then be done analytically. Combining the results and expanding for small η yields the desired expansion

$$\Delta\mathcal{S}_{\text{EE}}^{(1)} = \frac{t_0}{4G} \left[\frac{q}{4d} \ell^{d-1} V_{S^{d-1}} - \frac{B(\frac{2d-3}{2d-2}, \frac{d}{2d-2})}{8(d-2)} q^{\frac{d-2}{d-1}} \ell^{d-2} V_{S^{d-2}} + o(q^{\frac{d-2}{d-1}} \ell^{d-2}) \right]. \quad (37)$$

The volume of the n -dimensional unit sphere is $V_{S^n} = 2\pi^{(n+1)/2}/\Gamma(\frac{n+1}{2})$ and $B(a, b)$ is the Euler beta function. The first term is a volume term on the boundary, and could also be obtained by naïvely expanding the integrand in (26). But the naïve expansion then breaks down and gives divergent subleading terms. The noteworthy feature of the second term in (37) is that it scales as just an area, with no logarithmic enhancement. For the strip we can analogously expand the entanglement entropy for large q . The interesting part for the discussion below is the leading volume term, which reads

$$\Delta\mathcal{S}_{\text{EE}}^{(1)} = \frac{t_0}{4G} \frac{\ell_\star^2}{\ell^2} \frac{\pi}{d(d-1)} V_{d-2} \ell q + o(q\ell^{d-1}). \quad (38)$$

We see that for the disc and the strip the entanglement entropies for large ℓ are proportional to q and the volume of the region A , V_A . For the disc we have $V_A = \ell^{d-1} V_{B^{d-1}}$, with the volume of the unit ball $V_{B^n} = \pi^{n/2}/\Gamma(1 + \frac{n}{2})$, and for the strip $V_A = \ell V_{d-2}$. Using this to work out the coefficients, they turn out to be different. We will see this very explicitly in the entanglement thermodynamics discussion and come back to the interpretation afterwards.

a. Entanglement thermodynamics and universal behavior for small ℓ We now discuss our results in the context of the first law of entanglement thermodynamics proposed in [27]. A field-theory derivation has been given in [28]. This concept allows to define a notion of temperature

also in situations where the usual thermodynamic one is ill defined or trivial. We exploit this for our zero-temperature study. The entanglement temperature T_{ent} is defined by

$$T_{\text{ent}} \delta \mathcal{S}_{\text{EE}}(A) = \delta E(A) , \quad E(A) = \int_A \langle T_{tt} \rangle_{\text{CFT}} , \quad (39)$$

and we also define $\beta_{\text{ent}} = 1/T_{\text{ent}}$. To evaluate that definition we use (17) and (14), which yields

$$\delta E(A) = T_0 \alpha(d) V_A q^{\frac{1}{d-1}} \delta q , \quad (40)$$

where $V_A = \ell^{d-1} V_{\text{B}^{d-1}}$ for the sphere and $V_A = \ell V_{d-2}$ for the strip. For $q\ell^{d-1} \ll 1$ we then find from (35)

$$\beta_{\text{ent}}^{\ell \ll 1} = \frac{2\pi\ell}{d+1} , \quad \beta_{\text{ent}}^{\ell \ll 1} = \frac{\pi^{3/2} \Gamma(2 + \frac{1}{d-1}) \ell_*^2}{d \Gamma(\frac{3}{2} + \frac{1}{d-1}) \ell} , \quad (41)$$

for the sphere and the strip, respectively. This is exactly the shape-dependent but otherwise universal behavior predicted for small regions in [27]. It was interpreted as the fact that localized excitations of the vacuum state store a universal amount of information, regardless of their precise nature. We now turn to large regions, $q\ell^{d-1} \gg 1$. Due to the volume terms in (37), (38) and the fact that the energy density in (39) is itself proportional to the volume, the entanglement temperature settles on non-zero constant values for large ℓ . Concretely, we find for the sphere and the strip

$$\beta_{\text{ent}}^{\ell \gg 1} = \frac{2\pi^{3/2}}{dq^{1/(d-1)}} \frac{\Gamma(\frac{d+1}{2})}{\alpha(d)\Gamma(\frac{d}{2})} , \quad \beta_{\text{ent}}^{\ell \gg 1} = \frac{2\pi^{3/2}}{dq^{1/(d-1)}} \frac{\Gamma(\frac{1}{2d-2})}{\Gamma(\frac{d-2}{2d-2})\Gamma(\frac{d}{2d-2})^2} , \quad (42)$$

respectively. For intermediate sizes we can obtain β_{ent} numerically using (39) and (40) with (26) for the sphere and (34) for the strip. The result is illustrated in Fig. 1. Despite the complicated appearance of the entanglement entropies, the results just smoothly interpolate between our analytical results for small and large ℓ . In passing we also note that β_{ent} and thus also the renormalized entanglement entropies increase monotonically with ℓ for fixed q . This is consistent with the entanglement F-theorem for spherical regions in $d=3$ [29].

b. Shape-dependent volume terms and the entropy of the global state For small regions we have seen that our results reproduce the universal behavior predicted from entanglement thermodynamics, and we now come back to a more detailed discussion of large regions and the volume terms in (37), (38). In the vacuum state there are no extensive terms in the entanglement entropy. The correlations are local and the entropy of the reduced state ρ_A is mainly due to ignored correlations across the boundary of A . These are UV sensitive and the entropy thus divergent, which yields the usual area law [30]. The finite parts for the disc and the strip are then at most $\mathcal{O}(\ell^0)$. This changes when we consider a global state with non-vanishing entropy density, such as a thermal state. For large regions the finite part of the entanglement entropy is then dominated by the entropy of the global state. This contribution has a trivial shape dependence: for a translationally invariant system it is simply the entropy density of the global state multiplied by the volume of A . For our state of finite charge density the situation is not that different, as it also has a non-vanishing entropy density $s_0 = 4\pi T_0 q/d$, even at zero temperature [4]. We therefore should expect an extensive contribution in the finite parts, and it should have the trivial shape dependence $\Delta \mathcal{S}_{\text{EE}} = s_0 V_A + o(q\ell^{d-1})$. Interestingly enough, though, this is not what we find. The results of (37), (38) show that the volume terms have a non-trivial shape dependence: they can not be understood as a universal factor multiplying the volume of A . This can also be seen in

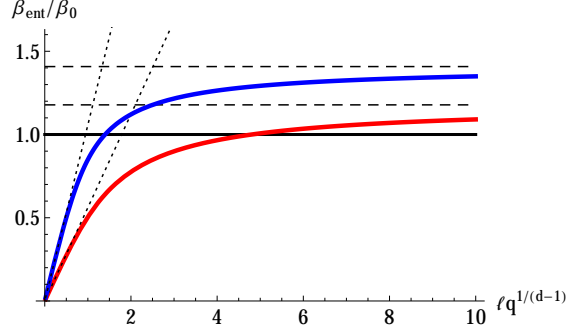


FIG. 1. The upper blue and lower red curves show $\beta_{\text{ent}} = 1/T_{\text{ent}}$ with the entanglement temperature defined in (39) for the strip and the disc, respectively, in $d=4$. Both are normalized to $\beta_0 = \delta s_0/\delta E$, which is the value we would expect for large regions if the entanglement entropy were dominated by the entropy of the global state. The ratio only depends on $q\ell^{d-1}$ for the sphere and the strip, and β_0 is indicated as solid black line. The dotted lines show the shape-dependent but otherwise universal behavior expected for small regions, which the curves reproduce nicely. For larger regions we see a smooth cross over and the curves saturate at the constant values (42) shown as dashed lines. The asymptotic values are different and greater than that expected due to the entropy of the global state in both cases, and the same applies for the entanglement entropy densities.

Fig. 1. If the volume terms had a trivial shape dependence, the inverse entanglement temperature for large regions would approach the universal value $\beta_0 = \delta s_0/\delta E$ regardless of the shape, which is clearly not the case. We also see that the inverse entanglement temperatures are both greater than β_0 , so we are led to conclude that there is a shape-dependent contribution in addition to the expected trivial part. This shape dependence can hardly be due to the entropy of the global state, which is translationally invariant and has no information on possible entangling surfaces. It therefore signals the presence of long-range correlations, which are not confined to the entangling surface but rather correlate the entire interior of the region A with its complement.

V. TRANSITION TO THERMAL ENTROPY AT FINITE TEMPERATURE

We now turn to the case of finite temperature and study the fate of the extensive entanglement correlations found in the previous section. If the shape-dependent volume terms indeed signal long-range correlations, we would expect that for an intermediate range of distances our previous results remain valid, but over long enough distances thermal fluctuations should wash them out eventually. For the renormalized entanglement entropy this means that, as we increase the size of the region A , the coefficient of the volume term should at some point decrease to the thermal entropy density. This does not imply that the entropy itself has to be non-monotonic. Indeed, we find a smooth cross over from an “entanglement-dominated phase” to a “thermal phase”, during which the entropy still grows monotonically with the size of A . We will focus on the strip, since we can then find the corresponding minimal surface at finite temperature in a decent form. We did not succeed in solving the Einstein equations analytically, but the problem can be formulated in a form which makes it easily accessible for numerical methods. We first derive the expression for the change in minimal area due to the backreaction and then discuss how to numerically solve (10).

As for $T=0$, we parametrize the codimension-2 minimal surface for the strip by $x^1 = x^1(z)$, and the induced metric then reads

$$\gamma_{\text{min}} = (g_{zz} + g_{xx}x'(z)^2) dz \otimes dz + g_{i_0 j_0} dx^{i_0} \otimes dx^{j_0}. \quad (43)$$

The integrand for the minimal area, $\sqrt{\gamma_{\min}}$, does not depend on $x^1(z)$ and we get a conserved quantity ℓ_* , parametrizing the extension of the minimal surface into the bulk. This yields

$$\frac{dx^1}{dz} = \frac{z\sqrt{g_{zz}}}{\sqrt{(\ell_*/z)^{2d-2} - 1}}. \quad (44)$$

The width ℓ of the strip is then obtained from its extension in the x^1 -direction as

$$\ell = 2x(0) = \int_0^{\ell_*} \frac{2dz}{\sqrt{b(z)((\ell_*/z)^{2d-2} - 1)}}. \quad (45)$$

We again only need the diagonal elements of $T_{\min}^{\mu\nu}$ and they take the form given in (29), now with γ_{\min} given in (43). The entanglement entropy correction (19) with (8) then becomes

$$\mathcal{S}_{\text{EE}}^{(1)} = \frac{t_0 V_{d-2}}{4G} \int_{\epsilon}^{\ell_*} dz \sqrt{\gamma_{\min}} \left[\frac{g_{zz} T^{zz}}{d(d-1)} + j(z) \left(\frac{z^{2d-2}}{\ell_*^{2d-2}} + d - 2 \right) \right]. \quad (46)$$

Somewhat remarkably, the only difference to (30) is in the volume form – the redshift factor drops out in the first term in square brackets and the second one does not include it either. We substitute $v = z/\ell_*$ and rewrite the expression as

$$\mathcal{S}_{\text{EE}}^{(1)} = \frac{t_0 V_{d-2}}{4G} \ell_*^{2-d} \int_{\epsilon/\ell_*}^1 dv \frac{v^{1-d}}{\sqrt{b(v\ell_*)(1 - v^{2d-2})}} \left[\frac{1}{d} + (j(v\ell_*) - c_2) (v^{2d-2} + d - 2) \right]. \quad (47)$$

The first term in the square brackets yields the zero-density result. We define a finite quantity by subtracting off the entanglement entropy in the vacuum state at zero charge and zero temperature

$$\Delta \mathcal{S}_{\text{EE}}^{(1)} = \mathcal{S}_{\text{EE}}^{(1)} - \mathcal{S}_{\text{EE},q=T=0}^{(1)}. \quad (48)$$

This is the expression we will evaluate numerically below. Before turning to the results, we discuss the limit of an asymptotically wide strip. In that case $\ell_* \rightarrow z_h$ and we can get a simple expression for the entropy density. The bulk minimal surface then approximately consists of two pieces: one wraps the part of the horizon corresponding to the strip, and the other connects that piece to the boundary. The latter just contributes a constant, and only the former is therefore relevant for the entropy density. The induced metric is simply $\gamma_{\min} = g_{ij} dx^i \otimes dx^j$ and we find

$$S_{\text{EE},\infty} = \frac{1}{4G} \int_{z=z_h} \sqrt{\gamma} = \frac{V_{d-1}}{4G} z_h^{1-d} (1 + t_0 j(z_h))^{(d-1)/2} + \mathcal{O}(V_{d-1}^0) + \mathcal{O}(t_0^2). \quad (49)$$

c. Numerical solution and results We now turn to the evaluation of the renormalized entropy (48). To this end we have to solve (10) for the backreaction, which can be done as follows. With the initial value (11) we can integrate (10b) to get \tilde{j} , and then solve (10a) for \tilde{h} . To further integrate these using (9), we have to fix two more constants of integration. Since we do not want to source the CFT energy-momentum tensor, we once again fix $h(0) = j(0) = c_2$. We note that the temperature of the backreacted solution is not necessarily the same as that of (2). This should not concern us at the moment, but will be relevant for the cross check with thermodynamic results below.

With the backreaction at hand, we can then straightforwardly evaluate (48) with (47). The results are illustrated in Fig. 2(a), where we have chosen $d = 4$ as appropriate for $\mathcal{N} = 4$ SYM. For small ℓ we see that the entanglement entropy agrees very well with the zero-temperature analysis. The figure also confirms our expectation for large ℓ : For $\ell \gg 1$ but still small enough such that ℓ_* is sufficiently far from the horizon, we have the shape-dependent contribution to the volume law.

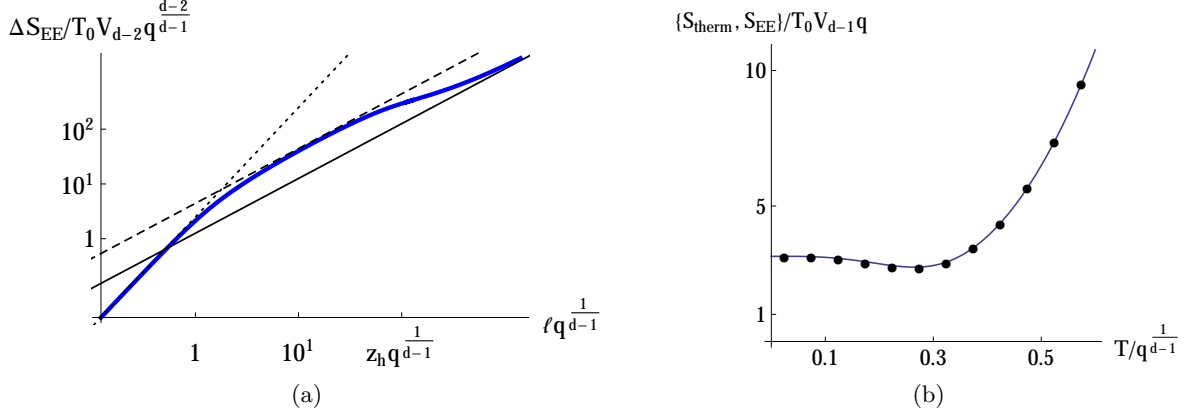


FIG. 2. The figure on the left hand side shows as blue solid curve the entanglement entropy of the strip, (48). The rescaling makes it a function of only the dimensionless quantities $q\ell^{d-1}$ and $T/q^{\frac{1}{d-1}}$. We have chosen $d=4$ and a small temperature such that $z_h q^{\frac{1}{d-1}} = 10^2$. The dotted and dashed lines directly correspond to those of Fig. 1: they show the zero-temperature results for the narrow and the wide strip, respectively. The black solid line shows the entropy for the infinitely extended strip, (49), which is expected to reproduce the entropy of the global state. That this is indeed the case is shown on the right hand side: the solid line shows the entropy of the global state, and the markers show the entanglement entropy for the asymptotically large strip as given in (51). We conclude that the entanglement entropy is well described by the zero-temperature result as long as the minimal surface does not extend all the way to the horizon. For even wider strips, the shape-dependent enhancement vanishes, leaving only the entropy of the global state.

At around the point where ℓ equals z_h , the minimal surface starts to really probe the horizon (note the linear relation between ℓ and ℓ_* for $T = 0$). As ℓ is increased further, the renormalized entropy finally settles on the universal volume contribution due to the entropy of the global state. This shows that the thermal fluctuations wash out the long-range correlations on scales greater than z_h .

As a cross check for our numerical implementation we explicitly verify that the asymptotic form of the entanglement entropy for the strip, (49), reproduces the entropy density of the global state. To this end we first have to determine the temperature of the perturbed solution, as usually by demanding that there should be no conical singularity in the Euclidean solution. To linear order in t_0 this yields

$$T = \frac{d}{4\pi z_h} \left[1 + t_0 \left(\frac{h(z_h) - c_2}{2} - \frac{\delta z_h}{z_h} \right) \right] + \mathcal{O}(t_0^2), \quad (50)$$

where we have defined the locus of the horizon in the perturbed solution as $z'_h = z_h + t_0 \delta z_h + \mathcal{O}(t_0^2)$. To keep the temperature unchanged we thus have to fix $\delta z_h = z_h(h(z_h) - c_2)/2$.

We can now compare the asymptotic behavior of the entanglement entropy to the corresponding entropy density of the global state. We expand (49) to linear order in t_0 , taking into account the shift in z_h to ensure that the perturbed solution keeps the original temperature. This yields

$$\mathcal{S}_{EE}^{(1)} = 2\pi T_0 V_{d-1} (d-1) z_h^{1-d} (j(z_h) - h(z_h) + c_2). \quad (51)$$

We compare this to the thermodynamic entropy $\mathcal{S} = V_{d-1}^{-1} (-\partial\Omega/\partial T)_{\mu, V_{d-1}}$. The thermodynamic potential Ω and chemical potential μ were given in [4] and the results, as illustrated in Fig. 2(b), agree nicely.

VI. DISCUSSION

In this work we have studied entanglement among the charge carriers in the quantum liquid phase of flavored $\mathcal{N} = 4$ SYM identified in [4]. A specific question in this context is whether there is a logarithmic enhancement of the area law signaling the presence of a Fermi surface. Given the non-vanishing ground state entropy and the somewhat peculiar scaling of the specific heat, the entanglement properties also add an interesting piece of information to characterize the phase, even where they are as expected. More generally speaking, our investigation provides an example where results on the entanglement entropy for a non-trivial state can be obtained in closed form. For the sphere and the strip we have given it in (26) and (34), respectively. For small regions, these follow the universal behavior predicted in [27, 28]. The more interesting feature are certainly the volume terms dominating the renormalized entanglement entropy for large regions. Their presence by itself is not very surprising in view of the non-vanishing entropy density of the global state². In fact, extensive terms were anticipated in a similar context already in [35, 36], where they were due to the fact that the bulk theory was capped off in the IR. However, both of these considerations would lead to the expectation of universal volume terms with a trivial shape dependence. What we have found, on the other hand, is that in addition to these expected terms there are contributions with non-trivial shape dependence. As the entropy of the translationally invariant global state can hardly account for the shape dependence, we conclude that these are actually a feature due to entanglement correlations. While extensive entanglement entropy seems to be a generic feature of random states in many-body systems [37], this is crucially different in local quantum field theories. The presence of only short-range interactions typically reduces the amount of entanglement to an area law for the entanglement entropy in the latter. The clear-cut implementation of the area law was in fact one of the immediate validations for the holographic prescription of [17]. Known exceptions are systems with a Fermi surface, where a logarithmically enhanced area law of the form $(k_F \ell)^{d-2} \log k_F \ell$ would be expected [38–40], and non-local theories [41–45]. In spin-chain models the presence of extensive terms for certain excited states could indeed be linked to the (non-)locality properties of the Hamiltonian for which the particular state would be a ground state [46]. Our setup, on the other hand, does not seem to straightforwardly fit into either of these categories. This certainly makes the high degree of entanglement among the charge carriers signaled by the extensive terms a remarkable feature. It immediately raises the question for the precise nature of the correlations, and may eventually help explain the departure from Fermi liquid behavior.

For finite temperature we have shown that the non-trivial shape dependence of the volume terms only persists up to a scale set by the inverse temperature. Beyond that scale the renormalized entanglement entropy settles on the shape-independent volume law contribution due to the entropy of the global state, as shown in Fig. 2. This seems nicely consistent with the interpretation that the shape-dependence of the volume terms is indeed due to long-range entanglement correlations, which on large enough scales are simply washed out by thermal fluctuations. We close with a more detailed discussion of Fermi surfaces. The indicator would be a logarithmically enhanced area contribution, and we would expect this feature for a Fermi liquid. Since we have considered a state of non-zero entropy density, one could have expected the enhancement of the area law to just be hidden behind the shape-independent extensive terms. The expansion (37), however, shows that the first subleading term already is a pure area term with no logarithmic enhancement. The area contribution is thus just enhanced by a shift in the coefficient. On the other hand, as argued above, the volume terms are non-trivial and an entanglement effect, so there actually is a more drastic enhancement instead. As discussed in Sec. II, the linearized approximation becomes unreliable in

² The state is different in that respect from those studied in [31, 32]. This also violates one of the assumptions in [33], and may explain why their argument for an at most logarithmic enhancement of the area law does not apply. Finally, we note that the finite entropy does not seem to signal an instability [34].

the deep IR limit. A conservative interpretation of our zero-temperature analysis therefore is as a limit where T is positive but small enough to be neglected on the scales on which we have studied entanglement correlations. This picture is confirmed by our finite-temperature analysis, showing that the volume terms are not just an artefact. Without the fully backreacted geometry we can, however, not draw definite conclusions on the limit $q\ell^{d-1} \rightarrow \infty$ and the presence of a Fermi surface.

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- [1] Juan Martin Maldacena, “The large N limit of superconformal field theories and supergravity,” *Adv. Theor. Math. Phys.* **2**, 231–252 (1998), [arXiv:hep-th/9711200](#).
 - [2] Edward Witten, “Anti-de Sitter space and holography,” *Adv. Theor. Math. Phys.* **2**, 253–291 (1998), [arXiv:hep-th/9802150](#).
 - [3] S. S. Gubser, Igor R. Klebanov, and Alexander M. Polyakov, “Gauge theory correlators from non-critical string theory,” *Phys. Lett.* **B428**, 105–114 (1998), [arXiv:hep-th/9802109](#).
 - [4] A. Karch, D.T. Son, and A.O. Starinets, “Zero Sound from Holography,” (2008), [arXiv:0806.3796 \[hep-th\]](#).
 - [5] F. Haldane and E. Rezayi, “Periodic Laughlin-Jastrow wave functions for the fractional quantized Hall effect,” *Phys.Rev.* **B31**, 2529–2531 (1985).
 - [6] X.G. Wen and Q. Niu, “Ground-state degeneracy of the fractional quantum Hall states in the presence of a random potential and on high-genus Riemann surfaces,” *Phys.Rev.* **B41**, 9377–9396 (1990).
 - [7] J. Shao, E.-A. Kim, F. D. M. Haldane, and E. H. Rezayi, “Entanglement entropy of the $\nu=1/2$ composite fermion non-Fermi liquid state,” *ArXiv e-prints* (2014), [arXiv:1403.0577 \[cond-mat.str-el\]](#).
 - [8] G Vidal, J.I. Latorre, E Rico, and A. Kitaev, “Entanglement in quantum critical phenomena,” *Phys.Rev.Lett.* **90**, 227902 (2003), [arXiv:quant-ph/0211074 \[quant-ph\]](#).
 - [9] Alexei Kitaev and John Preskill, “Topological entanglement entropy,” *Phys.Rev.Lett.* **96**, 110404 (2006), [arXiv:hep-th/0510092 \[hep-th\]](#).
 - [10] Michael Levin and Xiao-Gang Wen, “Detecting Topological Order in a Ground State Wave Function,” *Phys.Rev.Lett.* **96**, 110405 (2006).
 - [11] Horacio Casini, Marina Huerta, and Jose Alejandro Rosabal, “Remarks on entanglement entropy for gauge fields,” *Phys.Rev.* **D89**, 085012 (2014), [arXiv:1312.1183 \[hep-th\]](#).
 - [12] Andreas Karch and Emanuel Katz, “Adding flavor to AdS / CFT,” *JHEP* **0206**, 043 (2002), [arXiv:hep-th/0205236 \[hep-th\]](#).
 - [13] Shinpei Kobayashi, David Mateos, Shunji Matsuura, Robert C. Myers, and Rowan M. Thomson, “Holographic phase transitions at finite baryon density,” *JHEP* **0702**, 016 (2007), [arXiv:hep-th/0611099 \[hep-th\]](#).
 - [14] Andreas Karch and Andy O’Bannon, “Metallic AdS/CFT,” *JHEP* **0709**, 024 (2007), [arXiv:0705.3870 \[hep-th\]](#).
 - [15] Andreas Karch and Andy O’Bannon, “Holographic thermodynamics at finite baryon density: Some exact results,” *JHEP* **0711**, 074 (2007), [arXiv:0709.0570 \[hep-th\]](#).
 - [16] David Mateos, Shunji Matsuura, Robert C. Myers, and Rowan M. Thomson, “Holographic phase transitions at finite chemical potential,” *JHEP* **0711**, 085 (2007), [arXiv:0709.1225 \[hep-th\]](#).
 - [17] Shinsei Ryu and Tadashi Takayanagi, “Holographic derivation of entanglement entropy from AdS/CFT,” *Phys.Rev.Lett.* **96**, 181602 (2006), [arXiv:hep-th/0603001 \[hep-th\]](#).

- [18] Aitor Lewkowycz and Juan Maldacena, “Generalized gravitational entropy,” *JHEP* **1308**, 090 (2013), [arXiv:1304.4926 \[hep-th\]](#).
- [19] Francesco Bigazzi, Aldo L. Cotrone, Javier Mas, Daniel Mayerson, and Javier Tarrio, “D3-D7 Quark-Gluon Plasmas at Finite Baryon Density,” *JHEP* **1104**, 060 (2011), [arXiv:1101.3560 \[hep-th\]](#).
- [20] Francesco Bigazzi, Aldo L. Cotrone, and Javier Tarrio, “Charged D3-D7 plasmas: novel solutions, extremality and stability issues,” *JHEP* **1307**, 074 (2013), [arXiv:1304.4802 \[hep-th\]](#).
- [21] Konstantina Kontoudi and Giuseppe Policastro, “Flavor corrections to the entanglement entropy,” (2013), [arXiv:1310.4549 \[hep-th\]](#).
- [22] Han-Chih Chang and Andreas Karch, “Entanglement Entropy for Probe Branes,” *JHEP* **1401**, 180 (2014), [arXiv:1307.5325 \[hep-th\]](#).
- [23] Andreas Karch and Christoph F. Uhlemann, “Generalized gravitational entropy of probe branes: flavor entanglement holographically,” *JHEP* **1405**, 017 (2014), [arXiv:1402.4497 \[hep-th\]](#).
- [24] Sebastian de Haro, Sergey N. Solodukhin, and Kostas Skenderis, “Holographic reconstruction of space-time and renormalization in the AdS / CFT correspondence,” *Commun.Math.Phys.* **217**, 595–622 (2001), [arXiv:hep-th/0002230 \[hep-th\]](#).
- [25] Sean A. Hartnoll, Joseph Polchinski, Eva Silverstein, and David Tong, “Towards strange metallic holography,” *JHEP* **1004**, 120 (2010), [arXiv:0912.1061 \[hep-th\]](#).
- [26] Kristan Jensen and Andy O’Bannon, “Holography, Entanglement Entropy, and Conformal Field Theories with Boundaries or Defects,” *Phys.Rev.* **D88**, 106006 (2013), [arXiv:1309.4523 \[hep-th\]](#).
- [27] Jyotirmoy Bhattacharya, Masahiro Nozaki, Tadashi Takayanagi, and Tomonori Ugajin, “Thermodynamical Property of Entanglement Entropy for Excited States,” *Phys.Rev.Lett.* **110**, 091602 (2013), [arXiv:1212.1164](#).
- [28] Gabriel Wong, Israel Klich, Leopoldo A. Pando Zayas, and Diana Vaman, “Entanglement Temperature and Entanglement Entropy of Excited States,” *JHEP* **1312**, 020 (2013), [arXiv:1305.3291 \[hep-th\]](#).
- [29] H. Casini and Marina Huerta, “On the RG running of the entanglement entropy of a circle,” *Phys.Rev.* **D85**, 125016 (2012), [arXiv:1202.5650 \[hep-th\]](#).
- [30] Mark Srednicki, “Entropy and area,” *Phys.Rev.Lett.* **71**, 666–669 (1993), [arXiv:hep-th/9303048 \[hep-th\]](#).
- [31] Mohsen Alishahiha, M. Reza Mohammadi Mozaffar, and Ali Mollabashi, “Holographic Aspects of Two-charged Dilatonic Black Hole in AdS₅,” *JHEP* **1210**, 003 (2012), [arXiv:1208.2535 \[hep-th\]](#).
- [32] Manuela Kulaxizi, Andrei Parnachev, and Koenraad Schalm, “On Holographic Entanglement Entropy of Charged Matter,” *JHEP* **1210**, 098 (2012), [arXiv:1208.2937 \[hep-th\]](#).
- [33] Brian Swingle and T. Senthil, “Universal crossovers between entanglement entropy and thermal entropy,” *Phys.Rev.* **B87**, 045123 (2013), [arXiv:1112.1069 \[cond-mat.str-el\]](#).
- [34] Martin Ammon, Johanna Erdmenger, Shu Lin, Steffen Muller, Andy O’Bannon, *et al.*, “On Stability and Transport of Cold Holographic Matter,” *JHEP* **1109**, 030 (2011), [arXiv:1108.1798 \[hep-th\]](#).
- [35] Jose L.F. Barbon and Carlos A. Fuertes, “A Note on the extensivity of the holographic entanglement entropy,” *JHEP* **0805**, 053 (2008), [arXiv:0801.2153 \[hep-th\]](#).
- [36] Brian Swingle, “Entanglement Entropy at Finite Density from Extremal Black Holes,” (2009), [arXiv:0908.1737 \[hep-th\]](#).
- [37] Don N. Page, “Average entropy of a subsystem,” *Phys.Rev.Lett.* **71**, 1291–1294 (1993), [arXiv:gr-qc/9305007 \[gr-qc\]](#).
- [38] B. Swingle, “Entanglement Entropy and the Fermi Surface,” *Physical Review Letters* **105**, 050502 (2010), [arXiv:0908.1724 \[cond-mat.str-el\]](#).
- [39] Noriaki Ogawa, Tadashi Takayanagi, and Tomonori Ugajin, “Holographic Fermi Surfaces and Entanglement Entropy,” *JHEP* **1201**, 125 (2012), [arXiv:1111.1023 \[hep-th\]](#).
- [40] Liza Huijse, Subir Sachdev, and Brian Swingle, “Hidden Fermi surfaces in compressible states of gauge-gravity duality,” *Phys.Rev.* **B85**, 035121 (2012), [arXiv:1112.0573 \[cond-mat.str-el\]](#).
- [41] Jose L.F. Barbon and Carlos A. Fuertes, “Holographic entanglement entropy probes (non)locality,” *JHEP* **0804**, 096 (2008), [arXiv:0803.1928 \[hep-th\]](#).
- [42] Willy Fischler, Arnab Kundu, and Sandipan Kundu, “Holographic Entanglement in a Noncommutative Gauge Theory,” *JHEP* **1401**, 137 (2014), [arXiv:1307.2932 \[hep-th\]](#).
- [43] Joanna L. Karczmarek and Charles Rabideau, “Holographic entanglement entropy in nonlocal theories,” *JHEP* **1310**, 078 (2013), [arXiv:1307.3517 \[hep-th\]](#).
- [44] Noburo Shiba and Tadashi Takayanagi, “Volume Law for the Entanglement Entropy in Non-local

- QFTs,” *JHEP* **1402**, 033 (2014), [arXiv:1311.1643 \[hep-th\]](#).
- [45] Da-Wei Pang, “On holographic entanglement entropy of non-local field theories,” (2014), [arXiv:1404.5419 \[hep-th\]](#).
- [46] V. Alba, M. Fagotti, and P. Calabrese, “Entanglement entropy of excited states,” *Journal of Statistical Mechanics: Theory and Experiment* **10**, 20 (2009), [arXiv:0909.1999 \[cond-mat.stat-mech\]](#).